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Bilinear Holomorphic Differential Operators for the Jacobi Group

by

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Introduction

In the theory of (elliptic) modular forms two types of differential operators are quite familiar objects:

The (nonholomorphic) Maaß-operators

$$\delta_k = \frac{k}{2iy} + \frac{\partial}{\partial z}$$

$$\delta_k^r = \delta_{k+2r-2} \circ \cdots \circ \delta_{k+2} \circ \delta_k$$

and bilinear holomorphic differential operators

$$[f, g]_{k_1,k_2,v} = \sum_{\mu=0}^{v} (-1)^{v-\mu} \binom{v}{\mu} \frac{\Gamma(k_1+v)\Gamma(k_2+v)}{\Gamma(k+\mu)\Gamma(k_2+v-\mu)} \frac{\partial^\mu f \partial^{v-\mu} g}{\partial z^\mu \partial z^{v-\mu}}$$

(introduced by Cohen [7] and Kuznetzov [10] and sometimes called Cohen-brackets; for a generalization to $Sp(n, \boldsymbol{R})$ see [9]).

Both types of operators share some equivariance properties with respect to the action of $Sl_2(\boldsymbol{R})$ on functions defined on the upper half space $\boldsymbol{H}$ (they raise the weight by $2r$ and $2v$ respectively). There is a very simple connection between these two types of operators, which however cannot be found in the literature explicitly, namely an identity of type

$$f \times \delta_{k_2}^r g = \sum_{v=0}^{r} a_v \delta_{k_1+k_2+2v}^{r-v} [f, g]_{k_1,k_2,v}$$

with certain coefficients $a_v = a_{k_1,k_2,r,v}$. Such an identity (which holds for any (real) weights k_1, k_2 and any natural number v, provided that $k_1+k_2>0$) is an easy consequence of the theory of nearly holomorphic functions combined with the fact that the Cohen bracket operator is unique up to scalars. A more direct way to obtain the operators $[f, g]_v$ from the Maaß-operator $\delta_{k_2}^v$ is by considering the holomorphic projection of $f \times \delta_{k_2}^v g$ in the space of square-integrable functions on $\boldsymbol{H}$ with respect to $y^{k_1+k_2+2v-2} dxdy$. In this way one can also get a formula for the most interesting coefficient a_r in the equation above. If we assume that a_r is different from zero, then the equation above can be viewed as a way to construct $[\ ,\ ]_{k_1,k_2,\nu}$ from the Maaß operator $\delta_{k_2}^r$. Conversely we can recover the Maaß operator from the Cohen bracket because

$$y^{k_1}[y^{-k_1},\ -]_{k_1,k_2,r}$$

is up to a scalar equal to $\delta_{k_2}^r$. All these facts can be proved in a straightforward way and will be left to the reader.

The simple procedure above (some aspects of its also appear in the work of M. Harris, see [8] and subsequent papers of the same author) can of course be generalized to cases like symplectic groups or Jacobi groups. The advantage of this procedure is that it explains the existence of such bilinear holomorphic differential operators without writing them down explicitly (avoiding rather complicated combinatorial problems); furthermore we can try to deduce as much information as possible about the structure of such holomorphic bilinear differential operators from their connection with Maaß-operators (Maaß-operators are very natural objects because they can be explained by Lie-theory). In this paper we show the usefulness of the procedure sketched above in the case of the Jacobi group. In particular, we show that the vector space of bilinear holomorphic differential operators raising the weight by ν is in general of dimension equal to $\left[\frac{\nu}{2}\right]+1$.

The starting point for our paper was the work of Choie [4] who in very explicit form generalized Cohen's brackets to the Jacobi group and our observation that the space of bilinear operators is not one-dimensional in general. The relation between our work and Choie's will be explained in the last section.

We finally mention that quite recently Choie and Eholzer [5] generalized Choie's previous work by writing down explicitly a family of bilinear holomorphic differential operators; this family has in general the "right" dimension $1+\left[\frac{\nu}{2}\right]$.

Contents

§1. Maaß-operators

In this section we describe the Maaß operators acting on functions defined on $\boldsymbol{H}\times C$. For details on these operators and their relation to the Lie algebra of the Jacobi group we refer to the papers of R. Berndt [1, 3].

Elements of $\boldsymbol{H}\times\boldsymbol{C}$ will be denoted by

$$(\tau, z)=(u+iv, x+iy)\,, \qquad v>0\,.$$

The Jacobigroup $G^J(\boldsymbol{R})$ is the set of all triples

$$g=[M. X, \xi]\,, \quad M\in SL_2(\boldsymbol{R})\,, \quad X=(\lambda, \mu)\in\boldsymbol{R}^2\,, \quad \xi\in S^1$$

with the group law

$$[M, X, \xi]\cdot[M', X', \xi']=\left[MM', XM'+X', \xi\cdot\xi'\cdot\boldsymbol{e}\left(\det\begin{pmatrix} XM' \\ X' \end{pmatrix}\right)\right].$$

Here we write $\boldsymbol{e}_m(z)$ for $e^{2\pi imz}$ and we omit m if $m=1$. The Jacobi group acts on $\boldsymbol{H}\times\boldsymbol{C}$ by

$$[M, X, \xi]\langle(\tau, z)\rangle=\left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d}\right).$$

For an index m and a weight k (both k and m in $\boldsymbol{Z}$) we also have an action of $G^J(\boldsymbol{R})$ on functions defined on $\boldsymbol{H}\times\boldsymbol{C}$ given by

$$(f|_{k,m}g)(\tau, z)=\xi^m(c\tau+d)^{-k}\boldsymbol{e}_m\left(-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d}+\lambda^2\tau+2\lambda z+\lambda\mu\right)f(g\langle(\tau, z)\rangle)\,.$$

If g is of the special form $g=[1_2, (\lambda, \mu), 1]$ then we write $f|_m(\lambda, \mu)$ instead of $f|_{k,m}g$. On $\boldsymbol{H}\times\boldsymbol{C}$ there is a measure given by

$$\mu_{k,m}^2 dV$$

with

$$\mu_{k,m}=v^{k/2}e^{-2\pi m(y^2/v)}$$

and

$$dV=v^{-3}dxdydudv$$

such that for all $g\in G^J(\boldsymbol{R})$

$$\int_{\boldsymbol{H}\times\boldsymbol{C}}\phi|_{k,m}g(\tau, z)\overline{\psi|_{k,m}g(\tau, z)}\mu_{k,m}^2 dV=\int_{\boldsymbol{H}\times\boldsymbol{C}}\phi(\tau, z)\overline{\psi(\tau, z)}\mu_{k,m}^2 dV\,,$$

provided that the integrals in question converge.

We are mainly interested in two differential operators

$$Y=\partial+\frac{y}{v}4\pi im$$

and

$$X=X_k=d+\frac{y}{v}\partial+2\pi im\left(\frac{y}{v}\right)^2+\frac{k}{2iv}\,,$$

where we write d for $\frac{\partial}{\partial\tau}$ and ∂ for $\frac{\partial}{\partial z}$.

We may view X and Y as analogues of the Maaß operator δ_k from the introduction. These operators arise naturally from the Lie algebra $\boldsymbol{g}^J$ of $G^J(\boldsymbol{R})$, for details on this see the paper of Berndt [1] (he calls them $\tilde{X}_+$ and $\tilde{Y}_+$). These operators satisfy a commutation relation

$$X_{k+1}Y = YX_k$$

(by direct computation or by transition to the Lie algebra as in [1]) and they raise the weights by 1 and 2 respectively, i.e. for all C^∞-functions f on $\boldsymbol{H}\times\boldsymbol{C}$ and all $g\in G^J(\boldsymbol{R})$ we have

$$(Yf)|_{k+1,m}g = Y(f|_{k,m}g)$$
$$(X_kf)|_{k+2,m}g = X_k(f|_{k,m}g)\,.$$

By $\mathscr{M}(m, k, v)$, $v\in\boldsymbol{N}$ we denote the $\boldsymbol{C}$ vector space generated by the operators

$$\{X^\alpha_{k+\beta}Y^\beta \mid 2\alpha+\beta = v\}$$

where

$$Y^\alpha = Y\cdots Y$$

and

$$X^\beta_k = X_{k+2\beta-2}\cdots X_{k+2}X_k\,.$$

It is clear that

$$\dim\mathscr{M}(m, k, v) = 1 + \left[\frac{v}{2}\right].$$

The elements of this space will be called Maaß-operators in the sequel.

We shall be interested in the relation between these Maaß operators and holomorphic bilinear differential operators $\mathscr{D}$ acting on pairs (ϕ, ψ) of functions on $\boldsymbol{H}\times\boldsymbol{C}$

$$\mathscr{D}(\phi, \psi) = \sum_{i,j,i',j'} a_{i,j,i',j'}(d^i\partial^j\phi)(d^{i'}\partial^{j'}\psi)$$

satisfying

$$\mathscr{D}(\phi, \psi)|_{k_1+k_2+v,m_1+m_2}g = \mathscr{D}(\phi|_{k_1,m_1}g, \psi|_{k_2,m_2}g) \tag{1}$$

for all $g\in G^J(\boldsymbol{R})$; it is easily seen, using the action of elements of type $g=[\begin{pmatrix} a & 0\\ 0 & 1/a\end{pmatrix}$, $(0, 0), 1]$ that only summands with $2i+j+2i'+j' = v$ occur in $\mathscr{D}$.
The space of all such operators will be denoted by $Bil^{k_1,k_2}_{m_1,m_2}(v)$.

REMARK. We are of course mainly interested in integral weights k_1, k_2. For a better understanding of some phenomena, it is however quite useful, to allow (sometimes) arbitrary complex weights. To do this, we remark that there is a branch of $\log(c\tau+d)$ defined on $\boldsymbol{H}$. Our bilinear operators (as well as the Maaß-operators) can be defined for such general weights and do not depend on a choice of branch of log; for the equivariance properties (1) of the differential operators we have to choose the same branch of log to define the various powers of $c\tau+d$ occuring there.

§2. Nearly holomorphic functions

Throughout this section, we (tacitly) assume that m is an integer different from zero. A nearly holomorphic function on $\boldsymbol{H}\times\boldsymbol{C}$ is by definition a polynomial in $\frac{1}{v}$ and $\frac{y}{v}$ with holomorphic functions on $\boldsymbol{H}\times\boldsymbol{C}$ as coefficients. The degree of a nearly holomorphic function

$$\sum g_{\alpha,\beta}\left(\frac{1}{v}\right)^{\alpha}\left(\frac{y}{v}\right)^{\beta}$$

is

$$\mathrm{Max}\{2\alpha+\beta \mid g_{\alpha,\beta}\neq 0\}\,.$$

Using standard generators of $G^J(\boldsymbol{R})$ we see from the formulas

$$\left(\mathrm{Im}\left(-\frac{1}{\tau}\right)\right)^{-1}=\frac{\tau^2}{v}-2i\tau$$

$$\frac{\mathrm{Im}\left(\frac{z}{\tau}\right)}{\mathrm{Im}\left(-\frac{1}{\tau}\right)}=\tau\frac{y}{v}-iz$$

$$\frac{\mathrm{Im}(z+\lambda\tau)}{v}=\frac{y}{v}+\lambda$$

that $G^J(\boldsymbol{R})$ acts (via $|_{k,m}$) on the space of nearly holomorphic functions of degree $\leq r$.

We want to prove a **structure theorem** for the space $\mathscr{A}(r,k)$ of nearly holomorphic functions of degree $\leq r$. In the case of $SL_2(\boldsymbol{R})$ such a result is well known, see [12] (in fact Shimura created a theory of nearly holomorphic functions for all symmetric domains of hermitian type, see eg. [13]).

We start from a simple lemma from linear algebra:

LEMMA 2.1. *Let r and k be integers, $r\geq 1$, and $0\leq i\leq\left[\frac{r}{2}\right]$. Then there exists a polynomial*

$$\sum_{\alpha=0}^{[r-2/2]} a_\alpha \left(\frac{1}{v}\right)^\alpha \left(\frac{y}{v}\right)^{r-2-2\alpha}$$

and a constant c such that

$$\left(\frac{1}{v}\right)^i \left(\frac{y}{v}\right)^{r-2i} = X_{k-2}\left(\sum_\alpha a_\alpha \left(\frac{1}{v}\right)^\alpha \left(\frac{y}{v}\right)^{r-2-2\alpha}\right) + cY^r(1).$$

The polynomial as well as the constant c are uniquely determined.

Proof. We use the following facts:

FACT 1.

$$Y^r(1) = \sum_{\alpha=0}^{[r/2]} d_\alpha^{(r)} \left(\frac{1}{v}\right)^\alpha \left(\frac{y}{v}\right)^{r-2\alpha}$$

is a homogeneous polynomial of degree r, its coefficients $d_\alpha^{(r)}$ are given by the following recursion formula:

$$d_\alpha^{(r+1)} = 4\pi i m d_\alpha^{(r)} + \frac{r-2\alpha+2}{2i} d_{\alpha-1}^{(r)}.$$

This can be easily seen by induction on r; in the formula above we put $d_{-1}^{(r)} = 0$ and also $d_\alpha^{(r)} = 0$ if $\alpha > \left[\frac{r}{2}\right]$.

We need some more properties of these coefficients:

FACT 2. *All the $d_\alpha^{(r)}$ are elements of*

$$(4\pi m)^{r-\alpha} \times i^r \times \left(\frac{1}{2}\right)^\alpha \times \boldsymbol{Z}.$$

For the "highest" coefficient a stronger statement is true:

$$d_{[r/2]}^{(r)} \in (4\pi m)^{r-[r/2]} \times i^r \times \left(\frac{1}{2}\right)^{[r/2]} \boldsymbol{Z}_2^\times,$$

where $\boldsymbol{Z}_2^\times$ denotes the units in the ring of 2-adic integers.

These statements again can be proved by induction on r, checking separately the cases r even and r odd.

FACT 3.

$$X_{k-2}\left(\frac{1}{v}\right)^\alpha \left(\frac{y}{v}\right)^{r-2-2\alpha} =$$

$$\left(\frac{-\alpha+k-2}{2i}\right)\left(\frac{1}{v}\right)^{\alpha+1}\left(\frac{y}{v}\right)^{r-2-2\alpha} + 2\pi i m \left(\frac{1}{v}\right)^\alpha \left(\frac{y}{v}\right)^{r-2\alpha}.$$

Therefore we have to solve the equation

$$\left(\frac{1}{v}\right)^i\left(\frac{y}{v}\right)^{r-2i} = \sum_{\alpha=0}^{[r/2]}\left(\left(\frac{-\alpha+1+k-2}{2i}\right)a_{\alpha-1}+2\pi i m a_\alpha + d_\alpha^{(r)} c\right)\left(\frac{1}{v}\right)^\alpha\left(\frac{y}{v}\right)^{r-2\alpha}.$$

considered as a system of $\left[\frac{r}{2}\right]+1$ linear equations for the $\left[\frac{r}{2}\right]+1$ unknown a_α and c with $0 \leq \alpha \leq \left[\frac{r-2}{2}\right]$. (We put $a_\alpha = 0$ for $\alpha > \left[\frac{r-2}{2}\right]$ and $a_{-1}=0$.)

To simplify our notation, let us put (for the moment)

$$b_j = \frac{-j+k-1}{2i} \qquad (1 \leq j \leq n)$$

and $n = \left[\frac{r}{2}\right]$. The coefficient matrix of our system of linear equations then looks as follows:

$$\begin{pmatrix} 2\pi i m & 0 & 0 & \cdots & d_0^{(r)} \\ b_1 & 2\pi i m & 0 & \cdots & d_1^{(r)} \\ 0 & b_2 & 2\pi i m & 0 & d_2^{(r)} \\ & & & & \\ & 0 & b_{n-1} & 2\pi i m & d_{n-1}^{(r)} \\ & & 0 & b_n & d_n^{(r)} \end{pmatrix}.$$

The determinant of this matrix is a sum of products

$$\pm d_{t-1}^{(r)}(2\pi i m)^{t-1}\prod_{j=t}^{n} b_j \qquad (1 \leq t \leq n+1)\,. \tag{2}$$

All the b_j are elements of $\frac{1}{2i}\boldsymbol{Z}$, therefore the products above can easily be seen (use fact 2) to lie in

$$(2\pi m)^r \times i^{n+r} \times 2^{-t+1-n+r} \times \boldsymbol{Z}\,.$$

We want to show that in the case $t=n+1$ the lowest possible power of 2 indeed occurs in (2). In that case (2) is equal to

$$\pm(2\pi i m)^{[r/2]} \cdot d_{[r/2]}^{(r)}$$

and the second part of fact 2 shows that the lowest possible power of 2 indeed occurs. This implies that the determinant cannot be zero for $k \in \boldsymbol{Z}$. Therefore our system of linear equations has a unique solution.

REMARK. In general (i.e. for $k \in \boldsymbol{R}$ or $k \in \boldsymbol{C}$) we only obtain, that the determinant is a (non-constant!) polynomial function of k, therefore the determinant will be nonzero up to finitely many weights.

As a straightforward application of the lemma above we get

PROPOSITION 2.1. *There exist coefficients $a_{\alpha,i}$ and c_i with $0 \leq \alpha \leq \left[\frac{r-2}{2}\right]$ and $0 \leq i \leq \left[\frac{r}{2}\right]$ such that for all nearly holomorphic functions $F = \sum_{i=0}^{[r/2]} \left(\frac{1}{v}\right)^i \left(\frac{y}{v}\right)^{r-2i} g_i$ of degree r the following identity holds*

$$F = X_{k-2}\left(\sum_{\alpha,i} a_{\alpha,i}\left(\frac{1}{v}\right)^{\alpha}\left(\frac{y}{v}\right)^{r-2-\alpha} g_i\right) + Y^r\left(\sum_i c_i g_i\right) + (*)$$

where $(*)$ is a nearly holomorphic function of degree $\leq r-1$. The coefficients $a_{\alpha,i}$ and c_i are uniquely determined by this property. More precisely there exists uniquely determined coefficients $f_{i,\alpha,\beta,\gamma,\delta}$ (independent of the g_i) such that

$$(*) = \sum_i \sum_{2\alpha+\beta+2\gamma+\delta=r} f_{i,\alpha,\beta,\gamma,\delta}\left(\frac{1}{v}\right)^{\alpha}\left(\frac{y}{v}\right)^{\beta} d^{\gamma}\partial^{\delta} g_i$$

where the case γ and δ both equal to zero is excluded.

Using induction on r we obtain now the following

THEOREM 2.1 (Structure Theorem for nearly holomorphic functions). *Let k, m, r be integers with $m \neq 0$, $r \geq 1$.*

The are (universal) coefficients $u_{i,j,\gamma,\delta}^{(s,t)}$ such that for any nearly holomorphic function $F = \sum_{2i+j \leq r}\left(\frac{1}{v}\right)^i\left(\frac{y}{v}\right)^j g_{ij}$ degree $\leq r$ the following identity holds:

$$F = \sum_{2s+t \leq r} X_{k-2s}^{s} Y^t (C_{s,t})$$

with

$$C_{s,t} = C_{s,t}(g_{ij}) = \sum_{i,j} \sum_{2\gamma+\delta \leq r-2s-t} u_{i,j,\gamma,\delta}^{(s,t)} d^{\gamma}\partial^{\delta} g_{ij}\,.$$

The coefficients $u_{i,j,\gamma,\delta}^{(s,t)}$ are uniquely determined by this property.

SUPPLEMENT. The theorem above has some equivariance properties:

For $M \in G^J(\mathbf{R})$ we put

$$G = F|_{k,m} M = \sum_{2i+j \leq r}\left(\frac{1}{v}\right)^i\left(\frac{y}{v}\right)^j h_{ij}\,.$$

We have now two possibilities to express G using the structure theorem above. From the uniqueness statement we get for all s and t

$$C_{s,t}((g_{ij}))|_{k-2s-t,m} M = C_{s,t}((h_{ij}))\,.$$

§3. The first inequality

We start from the function $\hat{\mu}$, defined by

$$\hat{\mu}(\tau, z) = \mu_{k_1,m_1}^{-2}(\tau, z) = v^{-k_1} e^{4\pi m_1 (y^2/v)} .$$

This function transforms as follows under $G^J(\boldsymbol{R})$:

$$\hat{\mu}|_{k_1,m_1}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu), \xi\right]$$

$$= \overline{\xi^m (c\tau + d)^{k_1} e_{-m_1}\left(\frac{-c(z+\lambda\tau+\mu)^2}{c\tau+d} + \lambda^2\tau + 2\lambda z + \lambda\mu\right)}\hat{\mu} .$$

For $D \in Bil_{m_1,m_2}^{k_1,k_2}(v)$ we define the differential operator $\hat{D}$ by

$$\hat{D}\phi = \hat{\mu}^{-1} D(\hat{\mu}, \phi) .$$

Then $\hat{D}$ is a polynomial in $\frac{1}{v}$, $\frac{y}{v}$ and ∂, d satisfying ($g \in G^J(\boldsymbol{R})$)

$$\begin{aligned}(\hat{D}\phi)|_{k_2+v,m_2} g &= (\hat{\mu}|_{k_1,m_1} g)^{-1} D(\hat{\mu}, \phi)|_{k_1+k_2+v,m_1+m_2} g \\ &= (\hat{\mu}|_{k_1,m_1} g)^{-1} D(\hat{\mu}|_{k_1,m_1} g, \phi|_{k_2,m_2} g) = \hat{D}(\phi|_{k_2,m_2} g) .\end{aligned}$$

Therefore $\hat{D}$ is indeed a Maaß-operator (if $m_2 \neq 0$) as we shall show at the end of this section.

We claim that the linear mapping $D \mapsto \hat{D}$ from $Bil_{m_1,m_2}^{k_1,k_2}(v)$ to $\mathscr{M}(m_2, k_2, v)$ is injective:

Assume that D is different for zero and given by

$$D(\psi, \phi) = \sum_{2i+j+2i'+j'=v} a_{i,j,i',j'} (d^i \partial^j \psi)(d^{i'} \partial^{j'} \phi) .$$

We fix i' and j' and put $r = 2i' + j'$. We shall show more generally that

$$\sum_{2i \leq r} b_i d^i \partial^{r-2i} \hat{\mu}_{k,m} = 0$$

has only the trivial solution $b_0 = \cdots = b_{[r/2]} = 0$ if $m \neq 0$.

For the moment we allow k to be any complex number. It is easy to see that it is sufficient to consider the case $r = 2r'$ even and to show that the assumption $b_{r'} \neq 0$ implies that k lies in a certain finite set of half-integral (non integral) numbers.

Using the Taylor expansion of $\hat{\mu}$ we get

$$\begin{aligned}&\sum_{i=0}^{r'} b_i d^i \partial^{2r'-2i} \hat{\mu} \\ &= \sum_{i=0}^{r'} b_i \sum_{j=r'-i}^{\infty} (-1)^i \frac{\Gamma(k+i+j)}{\Gamma(k+j)} \frac{(4\pi m)^j}{j!} \frac{(2j)!}{(2j-2r'+2i)!} v^{-k-j-i} y^{2j-2r'+2i} \\ &= \sum_{j=0}^{\infty} \left(\sum_{i=0}^{r'} b_i (-1)^i \frac{\Gamma(k+r'+j)}{\Gamma(k+r'-i+j)} \frac{(4\pi m)^{j+r'-i}}{(j+r'-i)!} \frac{(2j+2r'-2i)!}{(2j)!} \right) v^{-k-r'-j} y^{2j} .\end{aligned}$$

Therefore for any fixed j, the inner sum has to be zero, which implies that the polynomial

$$\sum_{i=0}^{r'}(-1)^i(4\pi m)^{-i}b_i\prod_{1\le\alpha\le i}(X+k+r'-\alpha)\prod_{\substack{1\le\beta\le 2r'-2i\\ \beta\ \mathrm{odd}}}(2X+2r'-2i-\beta)$$

is the zero polynomial.

Using $X=-\frac{1}{2}$ we get

$$b_{r'}\prod_{1\le\alpha\le i}\left(-\frac{1}{2}+k+\frac{r}{2}-\alpha\right)=0$$

which, for $b_{r'}\neq 0$ is impossible unless k is among the half integers

$$\frac{1}{2},\ -\frac{1}{2},\ \cdots,\ -r'+\frac{3}{2}.$$

Therefore we have the following (first) inequality of dimensions:

THEOREM 3.1. *Suppose that m_1 and m_2 are different from zero and k_1, k_2 are arbitrary integers. Then*

$$\dim Bil^{k_1,k_2}_{m_1,m_2}(v)\le\dim\mathscr{M}(m_2,k_2,v).$$

It remains to show that $\hat{D}\in\mathscr{M}(m_2,k_2,v)$. One can show this by purely Lie-theoretic agruments as in the semisimple setting, as was kindly pointed out to the author by Berndt [2]. Within the framework of this paper, a simple proof goes as follows: By the theory of nearly holomorphic functions as described in the previous section (here we need the assumption $m_2\neq 0$), we get an identity

$$\hat{D}=\sum X^iY^j\mathscr{D}_{ij}$$

the $\mathscr{D}_{ij}$ being holomorphic differential operators with the property that

$$\mathscr{D}_{ij}(f|_{k_2,m_2}g)=(\mathscr{D}_{ij}f)|_{k_2+v-2i-j,m_2}g$$

for all functions $f\colon \boldsymbol{H}\times\boldsymbol{C}\to\boldsymbol{C}$ and all $g\in G^J(\boldsymbol{R})$. By Lemma 6.2 the $\mathscr{D}_{ij}$ have to be powers of the heat operator, because they are equivariant for the Heisenberg group. The heat operator however (and all its power except the trivial one) does not satisfy the property (1) for the SL_2-part of the Jacobi group. So in the identity above, only the terms with $\mathscr{D}_{ij}$ trivial survive, which proves the assertion.

§4. The second inequality

Let k_1 and m_1 be arbitrary integers and let δ be a fixed nonzero element of $\mathscr{M}(k_2,m_2,v)$; the theory of nearly holomorphic functions as sketched in section 2 implies (under the condition $m_1+m_2\neq 0$) an identity of type

$$\phi\times\delta\psi=\sum_{2i+j\le v}X^i_{k_1+k_2+v-2i}Y^j\circ B_{i,j}(\phi,\psi) \tag{3}$$

valid for all holomorphic functions ϕ, ψ on $\boldsymbol{H}\times\boldsymbol{C}$; the $B_{i,j}$ are elements of $Bil^{k_1,k_2}_{m_1,m_2}(\nu-2i-j)$, independent of ϕ, ψ. We claim that (under some conditions) $B_{0,0}$ is nonzero, in other words, we get an injective linear map from $\mathcal{M}(m_2, k_2, \nu)$ to $Bil^{k_1,k_2}_{m_1,m_2}(\nu)$.

We first remark that $\delta(v^s)$ is of type

$$\delta(v^s)=v^s\sum_{2\alpha'+\beta=\nu}c_{\alpha'}\left(\frac{1}{v}\right)^{\alpha'}\left(\frac{y}{v}\right)^{\beta} \tag{4}$$

i.e. $\delta(v^s)$ consists of linear combinations of monomials

$$v^s\left(\frac{1}{v}\right)^{\alpha}y^{\beta}$$

with $2(\alpha-\beta)+\beta=\nu$, where (at the moment) s can be an arbitrary complex number. Now we look at the function

$$\psi(\tau, z)=\boldsymbol{e}(a\tau+bz)$$

with a, b integers and consider the integral

$$\int_{\boldsymbol{H}\times\boldsymbol{C}/G^J(\boldsymbol{Z})_\infty}\overline{\psi\psi\times\delta(v^s)}\mu_{k,m}^2 dV \tag{5}$$

where

$$dV=v^{-3}dxdydudv\,,$$

$$k=k_1+k_2+\nu\,,\qquad m=m_1+m_2$$

and

$$G^J(\boldsymbol{Z})_\infty=\left\{[M, X, 1]\in G^J(\boldsymbol{Z})\,\middle|\,M=\pm\begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix}, X=(0, *)\right\}$$

Concerning convergence it is sufficient to look at integrals like

$$\int\overline{\psi\psi v^s\left(\frac{1}{v}\right)^{\alpha}y^{\beta}}\mu_{k,m}^2 dV$$

with $2(\alpha-\beta)+\beta=\nu$.

Such integrals easily reduce to real integrals

$$\int_0^\infty v^{k+s-3-\alpha}e^{-4\pi av}\int_{-\infty}^{\infty}e^{-4\pi by}e^{-4\pi m(y^2/v)}y^{\beta}dydv\,.$$

The integral over y equals

$$e^{\pi(b^2v/m)}\int_{-\infty}^{\infty}\left(y-\frac{bv}{2m}\right)^{\beta}e^{-4\pi m(y^2/v)}dy$$

$$=e^{\pi(b^2v/m)}\sum_{\gamma=0}^{\beta}\binom{\beta}{\gamma}\left(-\frac{bv}{2m}\right)^{\beta-\gamma}\int_{-\infty}^{\infty}y^{\gamma}e^{-4\pi m(y^2/v)}dy$$

$$=e^{\pi(b^2v/m)}\sum_{\gamma=0,\,\gamma\text{ even}}\binom{\beta}{\gamma}\left(-\frac{bv}{2m}\right)^{\beta-\gamma}\Gamma\left(\frac{\gamma+1}{2}\right)\left(\frac{4\pi m}{v}\right)^{-(\gamma+1)/2}.$$

(The convergence is clear, if we assume m to be positive.)

It remains to consider

$$\int_0^{\infty}v^{k+s-3-\alpha+\beta-\gamma+(\gamma+1)/2}e^{-\pi v(4a-b^2/m)}dv=\frac{\Gamma\left(k+s-\frac{3}{2}-\alpha+\beta-\frac{\gamma}{2}\right)}{\left(\pi\left(4a-\frac{b^2}{m}\right)^{k+s-(3/2)-\alpha+\beta-(\gamma/2)}\right)}$$

which is convergent for

$$k+\mathrm{Re}(s)-\frac{v}{2}>\frac{3}{2}$$

if $D=4am-b^2$ is positive. Our computations provide convergence for m and D positive and $\mathrm{Re}(s)$ sufficiently large. It remains to show that our integral is not identically zero. To see this, we view the integral (5) as a function of s, D, b: As a polynomial in b, this integral is of degree β_0, where β_0 is maximal among the indices β in (4) with $c_{\alpha'}\neq 0$, so we are done provided that $\delta(v^s)$ is always nonzero for nonzero δ.

Suppose that

$$0\neq\delta=\sum_{i=0}^{[v/2]}a_i Y^j X^i\,.$$

We easily see (e.g. by induction) that

$$X^i(v^s)=v^s\left\{\frac{s^i}{(2\sqrt{-1})^i}v^{-i}+\mathscr{R}\right\}$$

where $\mathscr{R}$, considered as a polynomial in s, is of degree less than i.

It remains to look at $Y^j(v^{s-i})$. Again by induction we see that

$$Y^j(v^{s-i})=v^{s-i}\times\mathscr{P}_j\left(\frac{1}{v},\frac{y}{v}\right)$$

where $\mathscr{P}_j$ is a polynomial independent of s and different from zero, provided that $m_2\neq 0$ (look at the different powers of $(m_2\pi)$, which occur in $\mathscr{P}$!).

Now let i_0 be the largest number with $a_{i0}\neq 0$. Then $\delta(v^s)$ is of the form

$$v^s\left\{\frac{a_{i_0}s^{i_0}}{(2-\sqrt{-1})^{i_0}}\,v^{-i_0}\mathscr{P}_{v-2i_0}\left(\frac{1}{v},\frac{y}{v}\right)+\mathscr{R}'\right\}$$

where $\mathscr{R}'$, considered as a polynomial in s is of degree less than i_0; in particular, $\delta(v^s)$ is different from zero for $\delta\neq 0$.

On the other hand we can compute the integral in question using (3):

$$\sum_{2i+j\leq\nu}\int_{\boldsymbol{H}\times\boldsymbol{C}/G^J(\boldsymbol{Z})_\infty}\psi\overline{X^iY^j\circ B_{ij}(\psi,v^s)}\mu_{k,m}^2dV\,.$$

A direct computation along the same lines as above shows that

$$\int_{\boldsymbol{H}\times\boldsymbol{C}/G^J(\boldsymbol{Z})_\infty}\psi\overline{X\left(v^s\left(\frac{1}{v}\right)^\alpha y^\beta\psi\right)}\mu_{k,m}^2dV \tag{6}$$

$$\int_{\boldsymbol{H}\times\boldsymbol{C}/G^J(\boldsymbol{Z})_\infty}\psi\overline{Y\left(v^s\left(\frac{1}{v}\right)^\alpha y^\beta\psi\right)}\mu_{k,m}^2dV=0 \tag{7}$$

for any nonnegative integers α and β, in other words, only the term $B_{0,0}$ can contribute to the integral, therefore we have proved the nonvanishing of $B_{0,0}$ (under certain conditions). We remark that the vanishing of (6) and (7) could also be proved in a more Lie-theoretic way by studying the adjoint operators of X and Y in a suitable Hilbert space.

Summarizing these results we get

THEOREM 4.1. *Under the assumptions $m_1+m_2>0$ and $m_2\neq 0$ we have*

$$\dim\mathscr{M}(m_2,k_2,\nu)\leq\dim Bil_{m_1,m_2}^{k_1,k_2}(\nu)\,.$$

§5. Conclusions

5.1. The results of the provious sections were obtained under some additional conditions concerning the weights and indices. Using generators of the Jacobi group it is easy to see (as in the example in section 5.2) that the coefficients of an element of $Bil_{m_1,m_2}^{k_1,k_2}(\nu)$ are solutions of a system of homogeneous linear equations whose coefficients depend polynomially on k_1, k_2, m_1, m_2. The results obtained so far imply that the space of solutions of this system of equations has dimension $1+\left[\frac{\nu}{2}\right]$ on a certain subset of $\boldsymbol{Z}^4$; this subset is Zariski-dense in $\boldsymbol{C}^4$.

By the usual reasoning we obtain from this (and from the first inequality)

THEOREM 5.1 (summarizing the main results of this paper). *For any (complex) weights k_1 and k_2 and any indices m_1, m_2 we have*

$$1+\left[\frac{\nu}{2}\right]\leq\dim Bil_{m_1,m_2}^{k_1,k_2}(\nu)\,;$$

equality holds at least under the additional conditions $m_1\neq 0$, $m_2\neq 0$ and k_1 and k_2 integral.

5.2. We show by an example that the case of a strict inequality actually occurs: We start from an arbitrary element $\mathscr{D}\in Bil_{m_1,m_2}^{k_1,k_2}(2)$.

Such a $\mathscr{D}$ is of the form

$$\mathscr{D}(\phi_1, \phi_2)=A\phi_1(d\phi_2)+B(d\phi_1)\phi_2+C(\partial\phi_1)(\partial\phi_2)+D(\partial^2\phi_1)\phi_2+E\phi_1(\partial^2\phi_2)$$

with certain coefficients A, B, C, D, E. It is sufficient to study the equivariance properties of $\mathscr{D}$

$$\mathscr{D}(\phi_1, \phi_2)|_{k_1+k_2+2,m_1+m_2}g=\mathscr{D}(\phi_1|_{k_1,m_1}g, \phi_2|_{k_2,m_2}g)$$

for test functions of type

$$\phi_1=e^{a\tau+bz} \qquad a, b\in \boldsymbol{C}$$

$$\phi_2=e^{c\tau+dz} \qquad c, d\in \boldsymbol{C}$$

and for special elements $g\in G^J(\boldsymbol{R})$ of type

$$g(\lambda)=[1_2, (\lambda, 0), 1], \qquad \lambda\in\boldsymbol{R}$$

and

$$I=\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (0, 0), 1\right].$$

These elements, together with the "translations" $[\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, (0, 0), 1]$, $s\in\boldsymbol{R}$, $[1_2, (0, \mu), 1]$, $\mu\in\boldsymbol{R}$ and the elements of type $[1_2, (0, 0), \xi]$, $\xi\in S^1$ generate the group $G^J(\boldsymbol{R})$.

Straightforward calculations show that the equivariance properties for those test functions and the special elements $g(\lambda)$ are equivalent to the following equation

$$\begin{aligned} Ac+Ba+Cbd+Db^2+Ed^2 &= A(2\pi i m_2\lambda^2+c+d\lambda)+B(2\pi i m_1\lambda^2+a+b\lambda) \\ &\quad +C(2\pi i 2m_1\lambda+b)(2\pi i 2m_2\lambda+d) \\ &\quad +D(2\pi i 2m_1\lambda+b)^2+E(2\pi i 2m_2\lambda+d)^2 . \end{aligned}$$

The element I gives the equation

$$\begin{aligned} &(Ac+Ba+Cbd+Db^2+Ed^2)\times\left(\frac{1}{\tau}\right)^{k_1+k_2+2} \\ &= A\tau^{-k_1-k_2}\left\{\frac{-k_2}{\tau}+\frac{1}{\tau^2}2\pi i m_2 z^2+\frac{1}{\tau^2}(c-dz)\right\} \\ &\quad +B\tau^{-k_1-k_2}\left\{\frac{-k_1}{\tau}+\frac{1}{\tau^2}2\pi i m_1 z^2+\frac{1}{\tau^2}(a-bz)\right\} \\ &\quad +C\tau^{-k_1-k_2}\left\{\frac{-2\pi i m_1 2z}{\tau}+\frac{b}{\tau}\right\}\left\{\frac{-2\pi i m_2 2z}{\tau}+\frac{d}{\tau}\right\} \\ &\quad +D\tau^{-k_1-k_2}\left\{\frac{(-2\pi i m_1 2z+b)^2}{\tau^2}-\frac{4\pi i m_1}{\tau}\right\} \\ &\quad +E\tau^{-k_1-k_2}\left\{\frac{(-2\pi i m_2 2z+d)^2}{\tau^2}-\frac{4\pi i m_2}{\tau}\right\}. \end{aligned}$$

These two equations can be transformed into a system of linear equations for A, B, C by considering the first one as a polynomial in λ and the second one as a polynomial in $\frac{1}{\tau}$, z, b, d. The corresponding coefficient matrix for this system of 7 equations and 5 variables is

$$\begin{pmatrix} m_2 & m_1 & 8\pi i m_1 m_2 & 8\pi i m_1^2 & 8\pi i m_2^2 \\ 1 & 0 & 4\pi i m_1 & 0 & 8\pi i m_2 \\ 0 & 1 & 4\pi i m_2 & 8\pi i m_1 & 0 \\ k_2 & k_1 & 0 & 4\pi i m_1 & 4\pi i m_2 \\ m_2 & m_1 & 8\pi i m_1 m_2 & 8\pi i m_1^2 & 8\pi i m_2^2 \\ 1 & 0 & 4\pi i m_1 & 0 & 8\pi i m_2 \\ 0 & 1 & 4\pi i m_2 & 8\pi i m_1 & 0 \end{pmatrix}.$$

It is sufficient to consider the second, third and fourth row. We immediately see that the rank r of this matrix is at least two (and at most three). It is certainly equal to three if $m_1 m_2(m_1+m_2)\neq 0$. If this condition is not satisfied, then we have the following situation:

First case. $m_1 m_2 \neq 0$, $m_1+m_2=0$: The rank is two iff $k_1=k_2=\frac{1}{2}$.

Second case. $m_1\neq 0$, $m_2=0$: The rank is two iff $k_1=\frac{1}{2}$ and $k_2=0$.

Third case. $m_1=0$, $m_2\neq 0$: The rank is two iff $k_1=0$ and $k_2=\frac{1}{2}$.

Fourth case. $m_1=m_2=0$: The rank is always two.

5.3. We return to our standard situation (i.e. m_1 and m_2 both non-zero integers and k_1, k_2 arbitray integers). To each $D\in Bil^{k_1,k_2}_{m_1,m_2}(v)$, given by

$$D(\psi,\phi)=\sum_{2i+j+2i'+j'=v} a_{i'j'ij}(d^{i'}\partial^{j'}\psi)(d^i\partial^j\phi)$$

we associate a polynomial of two variables by

$$P=P_D(T_1,T_2)=\sum_{2i+j=v} a_{00ij}T_1^i T_2^j\,.$$

We claim that this linear mapping is injective: If P_D is zero, then the Maaß operator $\hat{D}$ associated to D as in section 2 has the property that all its summands of type $d^i\partial^j$ with $2i+j=v$ are zero; it is easily seen that such a Maaß-operator must be zero, hence D itself is zero.

This observation has some interesting application to Rankin convolutions of Jacobi forms, which we want to mention here: Going the other way around we may start from an arbitrary polynomial $P=P(T_1,T_2)=\sum_{2i+j=v}a_{ij}T_1^iT_2^j$ and associate to it an elment $D_P=D\in Bil^{k_1,k_2}_{m_1,m_2}(v)$ such that $P_D=P$. Let now f be a Jacobi cusp form of weight k and index m, g a Jacobi form of weight k_2, index m_2 and E the standard

Jacobi Eisenstein series of weight k_1 and index m_1. If $m=m_1+m_2$, $k=k_1+k_2+\nu$ we can consider the scalar product

$$\langle f, D_P(E, g)\rangle .$$

Using the same kind of argument as in [6] we can show (under some conditions on the weights k_1 and k_2) that up to some factors the scalar product above is equal to

$$\sum_{n\geq 1, r\in \boldsymbol{Z}, r^2\leq 4m_2 n} \frac{P_D(n, r)a(n, r)\overline{b(n, r)}}{(4mn-r^2)^{k-3/2}}$$

where $a(n, r)$ and $b(n, r)$ denote the Fourier coefficients of f and g. This generalizes the results in [6], where a special type of differential operator D (see also section 6 of the present paper) is considered with

$$P_D=(4m_2T_1-T_2^2)^{\nu/2} \quad (\nu \text{ even}) .$$

§6. Relation to Choie's work

Choie [4] found some very explicit element in $Bil^{k_1,k_2}_{m_1,m_2}(2r)$ in terms of heat operators $(j=1, 2)$

$$L_{m_j}=8\pi i m_j d-\partial^2$$

following quite closely the lines of Cohen's original work. Here we present a different approach to her operators, in particular we show that her operators are in some sense distinguished elements of $Bil^{k_1,k_2}_{m_1,m_2}(2r)$.

PROPOSITION 6.1. *Let m_1 and m_2 be non-zero integers. Then*

$$\sum_{i,j} a_{ij} L^i_{m_1}\otimes L^j_{m_2}$$

defines an element of $Bil^{k_1,k_2}_{m_1,m_2}(2r)$ if and only if

$$\sum_{i,j} m_1^i m_2^j a_{ij} d^i\otimes d^j$$

is an element of $Bil^{k_1-1/2,k_2-1/2}_{\text{elliptic}}(2r)$.

The notation used in this proposition is as follows:

$$(L^i_{m_1}\otimes L^j_{m_2})(\phi, \psi)(\tau, z):=(L^i_{m_1}\phi)(\tau, z)(L^j_{m_2}\psi)(\tau, z)$$

for functions ϕ and ψ on $\boldsymbol{H}\times \boldsymbol{C}$ and

$$(d^i\otimes d^j)(f, g)(\tau)=\left(\frac{\partial^i}{\partial\tau^i} f\right)(\tau)\left(\frac{\partial^j}{\partial\tau^j} g\right)(\tau)$$

for functions f and g on $\boldsymbol{H}$. Furthermore $Bil^{k_1-1/2,k_2-1/2}_{\text{elliptic}}(2r)$ is the set of bilinear differential operators acting on (pairs of)) functions defined on $\boldsymbol{H}$ and changing weights from $(k_1-\frac{1}{2}, k_2-\frac{1}{2})$ to k_1+k_2-1+2r.

The proof of this proposition follows directly from the following lemma:

LEMMA 6.1. *For any holomorphic function f on $\boldsymbol{H}$ and any complex number λ we define a holomorphic function $\tilde{f}=\tilde{f}_\lambda$ on $\boldsymbol{H}\times\boldsymbol{C}$ by*

$$\tilde{f}_\lambda(\tau, z)=f(\tau)e_m(\lambda^2\tau+2\lambda z)\,.$$

Then for any nonnegative integer j and any $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in Sl_2(\boldsymbol{R})$ we have

$$(c\tau+d)^{-1/2}d^j\left(f|_{k-1/2}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\times\mathscr{E}=\left(\frac{1}{8\pi im}L_m\right)^j\left(\tilde{f}_\lambda|_{k,m}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

where

$$\mathscr{E}=\mathscr{E}_\lambda\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\tau, z\right)=e_m\left(-\frac{cz^2}{c\tau+d}+\lambda^2\frac{a\tau+b}{c\tau+d}+2\lambda\frac{z}{c\tau+d}\right).$$

REMARK. a) The equality above is evident in the case of the identity matrix.

b) We should point out that if we sum up over all $\tilde{f}_\lambda$, $\lambda\in\boldsymbol{Z}$ we get the product of f with a Jacobi theta series.

It is sufficient to prove this lemma for $\lambda\in\boldsymbol{R}$; using the fact that the action of the Heisenberg group of functions on $\boldsymbol{H}\times\boldsymbol{C}$ and the heat operator commute with each other and using the equations

$$\tilde{f}_\lambda=\tilde{f}_0|_{k,m}[(1_2, (\lambda, 0), 1]$$

$$[1_2, (\lambda, 0), 1]\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (0, 0), 1\right]=\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (a\lambda, b\lambda), 1\right]$$

and

$$\mathscr{E}_\lambda\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)=\mathscr{E}_0\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}0\right)|_{k,m}[1_2, (a\lambda, b\lambda), 1]$$

we may even restrict to the case $\lambda=0$.

Now we recall that both d and $\frac{1}{8\pi im}L_m$ fail in the same way to commute with the action of $Sl_2(\boldsymbol{R})$:

$$d\left(h|_{k-1/2}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)=h'|_{k-1/2+2}\begin{pmatrix} a & b \\ c & d \end{pmatrix}+c\left(-k+\frac{1}{2}\right)f|_{k-1/2+1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for any function h on $\boldsymbol{H}$ and

$$L_m\left(\phi|_{k,m}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)=(L_m\phi)|_{k+2,m}\begin{pmatrix} a & b \\ c & d \end{pmatrix}+c8\pi im\left(-k+\frac{1}{2}\right)\phi|_{k+1,m}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for any function ϕ on $\boldsymbol{H}\times\boldsymbol{C}$.

By induction on j we get from this

$$d^j\left(h|_{k-1/2}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)=\sum_{\nu=0}^{j} A_{j,\nu}h^{(\nu)}|_{k-1/2+j+\nu}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and

$$\left(\frac{1}{8\pi im}L\right)^j\left(\phi|_{k,m}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)=\sum_{\nu=0}^{j} A_{j,\nu}\left(\left(\frac{1}{8\pi im}L_m\right)^{\nu}\phi\right)|_{k+j+\nu,m}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with the same coefficients $A_{j,\nu}$ depending on k and c. From these statements and using

$$\left(\frac{1}{8\pi im}L_m\right)^{\nu}\tilde{f}_0=d^{\nu}f$$

we get the lemma for $\lambda=0$.

Let us denote by $Bil_{m_1,m_2}^{k_1,k_2}(2r)^{\text{heat}}$ the subspace which can be written in terms of (polynomials in) heat operators. From the considerations above we get

THEOREM 6.1. *Let m_1, m_2 be non-zero integers. Then we have the following isomorphism*

$$Bil_{m_1,m_2}^{k_1,k_2}(2r)^{\text{heat}}\simeq Bil_{\text{elliptic}}^{k_1-1/2,k_2-1/2}(2r)$$

in particular we get from this that the space of bilinear differential operators considered by Choie is in fact one-dimensional in general.

There is another characterization of Choie's differential operators avoiding the (explicit) use of the heat operator. It is based on the following (propably well-known?)

LEMMA 6.2. *Let m be a non-zero integer and let*

$$P=\sum_{j=0}^{[r/2]} a_j d^j \partial^{r-2j}$$

be a differential operator acting on functions on $\boldsymbol{H}\times\boldsymbol{C}$ and satisfying

$$(Pf)|_m(\lambda,\mu)=P(f|_m(\lambda,\mu))$$

for all f and all $(\lambda,\mu)\in\boldsymbol{R}^2$.

Then P is zero unless r is even; for even r we have

$$(8\pi im)^{r/2}P=a_{r/2}L_m^{r/2}\,.$$

Proof. We consider as a test function

$$f=f_t(\tau,z)=e(t\tau)\,.$$

Then

$$P(f_t|_m(\lambda,0))(\tau,z)=\sum_{j=0}^{[r/2]} a_j(2\pi im\lambda^2+2\pi it)^j(4\pi im\lambda)^{r-2j}e((m\lambda^2+t)\tau+2m\lambda z)\,.$$

We compare this with

$$P(f_t)|_m(\lambda, 0) = a_{r/2}(2\pi i t)^{r/2} e((m\lambda^2 + t)\tau + 2m\lambda z)\,.$$

(This is true for r even; for r odd we have $P(f_t) = 0$.) Both expressions are of type

$$\text{polynomial in } t \times e(2(m\lambda^2 + t)\tau + 2m\lambda z)\,.$$

We get a system of linear equations for the a_j by comparing the coefficients of those polynomials in t. It is easy to see that all the a_j are zero if r is odd and that the a_j are uniquely determined by $a_{r/2}$ if r is even; in other words, P is zero if r is odd or it lies in a one-dimensional space and is therefore a multiple of $L_m^{r/2}$.

REMARK. Obviously there is also a non-homogeneous version of the lemma above.

Each operator $D \in Bil_{m_1,m_2}^{k_1,k_2}(v)$, given by

$$D(\phi, \psi) = \sum_{\alpha,\beta,\gamma,\delta} a_{\alpha\beta\gamma\delta}(d^\alpha \partial^\beta \phi)(d^\gamma \partial^\delta \psi)$$

arises in the obvious way (diagonalization) from an operator $\tilde{D}$ given by

$$\tilde{D}(\phi, \psi)(\tau, z, \tau', z') = \sum_{\alpha,\beta,\gamma,\delta} a_{\alpha\beta\gamma\delta}(d^\alpha \partial^\beta \phi)(\tau, z)(d^\gamma \partial^\delta \psi)(\tau', z')\,.$$

Now we define a subspace $Bil_{m_1,m_2}^{k_1,k_2}(2r)^+$ of $Bil_{m_1,m_2}^{k_1,k_2}(2r)$ which consists of those D with the extra properties

$$\tilde{D}(\phi, \psi)|_m(\lambda, \mu)^{(1)} = \tilde{D}(\phi|_m(\lambda, \mu), \psi)$$

and

$$\tilde{D}(\phi, \psi)|_m(\lambda, \mu)^{(2)} = \tilde{D}(\phi, \psi|_m(\lambda, \mu))$$

for all $(\lambda, \mu) \in \boldsymbol{R}^2$. The upper indices (1) and (2) indicate the type of variables on which we have to act. The lemma above about the heat operator implies then

THEOREM 6.2. *For non zero integers m_1, m_2 we have*

$$Bil_{m_1,m_2}^{k_1,k_2}(2r)^{\text{heat}} = Bil_{m_1,m_2}^{k_1,k_2}(2r)^+\,.$$

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